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ABSTRACT

The relationship between expert teachers' classroom behavior and their subject matter knowledge is an area of research that has not been examined extensively. To begin that study, one topic, fraction knowledge, was explored in depth as it occurred in natural teaching settings. Fractions are one of the more difficult topics in elementary arithmetic; much of this difficulty is attributable to the complex relationships among the meanings and representations of fractions and basic arithmetic operations. Despite these difficulties associated with fractions, many teachers are quite proficient at teaching children to perform operations with fractions. The content and organization of expert teachers' knowledge of fractions was investigated using interviews, card sorting tasks, and transcriptions of videotaped lessons. Semantic networks that reflected their knowledge of fractions were developed for individual teachers. Comparisons of these semantic networks showed that there were wide disparities among the knowledge of expert teachers. Some teachers displayed relatively rich conceptual knowledge of fractions while others relied upon precise knowledge of algorithms. Implications of these knowledge differences are discussed. (Author/JN)

Expertise in Mathematics Instruction: Subject Matter Knowledge

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Abstract

The relationship between expert teachers' classroom behavior and their subject matter knowledge is an area of research that has not been examined extensively. To begin that study, one topic, fraction knowledge, was explored in depth as it occurred in natural teaching settings. Fractions are one of the more difficult topics in elementary arithmetic; much of this difficulty is attributable to the complex relationships among the meanings and representations of fractions and basic arithmetic operations. Despite these difficulties associated with fractions, many teachers are quite proficient at teaching children to perform operations with fractions. We examined expert teachers' knowledge by using extensive protocols to investigate the content and organization of teachers' knowledge of fractions. These protocols involved interviews, card sorting tasks, and transcriptions of videotaped lessons. Semantic networks that reflected their knowledge of fractions were developed for individual teachers. Comparisons of these semantic networks showed that there were wide disparities among the knowledge of expert teachers. Some teachers displayed relatively rich conceptual knowledge of fractions while others relied upon precise knowledge of algorithms. Implications of these knowledge differences are discussed.

Expertise in Mathematics Instruction: Subject Matter Knowledge

This paper explores the organization and content of subject matter knowledge used by expert arithmetic teachers. Teaching can be considered a cognitive skill and, as such, it is amenable to analysis in ways similar to other cognitive skills (Leinhardt & Greeno, in preparation). The expertise involved in the cognitive aspects of teaching can be seen as emerging from two core-areas of knowledge: lesson structure and subject matter. Lesson structure knowledge includes the skills needed to plan and run a lesson smoothly, to pass easily from one segment to another, and to explain material clearly. Subject matter knowledge includes conceptual understanding, the particular algorithmic operations, the connection between different algorithmic procedures, the subset of the number system being drawn upon, understanding of classes of student errors, and curriculum presentation. Subject matter knowledge serves to support lesson structure and acts as a resource in the selection of examples, formulation of explanations, etc. Subject matter knowledge also constrains lesson structure in that the content of the lesson strongly influences how it is to be taught. The skills associated with lesson structure and subject matter knowledge are obviously intertwined. However, while it is unlikely that a teacher could be devoid of competence in one area and still be an expert, there seem to be cases in which teachers with similar outcomes or success levels have quite a different balance of skills. The objective of this work is to explore the nature, level, and utilization of subject matter knowledge among a set of expert teachers. It is compatible with a second line of research on lesson structure knowledge (Leinhardt, 1983b; Leinhardt & Greeno, in preparation).

Theoretical F rk

This research explores the dimensions, org and content of teacher's subject matter knowledge in one particular area, fractio ocused on fractions because of its importance in fourth grade mathematics. There veral significant algorithms to be taught in fourth grade: equivalent fractions, raising fractions to a specific denominator, reducing fractions, adding and subtracting with like and unlike denominators, mixed numbers, and converting mixe umbers to fractions and back again. Fractions are difficult to teach and to learn because they have several different conceptual meanings; a part of a regional whole, a portion of a discrete set of objects, a measurement point on a number line, or one number divided by another. They are difficult also because two numbers are used to represent a single quantity and because different number names can represent the same quantity. There is considerable evidence that children, even those who perform traditional tasks well, have quite primitive notions of the underlying concepts of fractions (Wachsmuth, Behr & Post, 1983).

While algorithmic competency of teachers is generally assumed (although observation indicates even this to be limited), the ability to represent the elements of an algorithm in a communicable way as with blocks or pictures and then connect the procedure and representation together is not assumed, nor is it frequently observed. (Resnick, 1982; Champagne & Rogalska-Saz, 1984). Given the importance of early mathematics instruction, it is surprising that so little research has addressed the issue of the type and level of subject matter skill used and required by teachers. With few exceptions (Coleman's vocabulary test, for example) research in the subject matter knowledge (level, organization, and understanding) of teachers has been alluded to but not studied (Evertson, Emmer, & Brophy, 1980; Pigge, Gibney, & Ginther, 1980). This is true both in process product research and in research in the cognitive process of teachers. The research reported here begins that study.

The content of the teacher's lesson is seen as the product of a cognitive system that represents knowledge in both declarative and procedural forms (Anderson, 1983). Declarative knowledge consists primarily of the facts that are known about a particular domain while procedural knowledge represents the algorithms and heuristics that operate on those facts. For example, basic multiplication facts can be stored in declarative form to be used procedurally when used to raise fractions to a common denominator. A common method for representing declarative knowledge is in the form of semantic networks (Woods, 1975). A semantic network is a node-link structure in which concepts are represented as nodes that are linked together according to a defined set of relationships. A common method for representing procedural knowledge is through the use of production rules that specify actions and the conditions under which those actions will be performed (Anderson, 1983).

To review, the overall cognitive system of a teacher is based upon at least two organized knowledge bases. One consists of general teaching skills and strategies, the other consists of specific information necessary for the content presentation. This second body of information has as resources the text material, teacher's manuals, and elements of experience that identify what's hard to teach. It also includes algorithmic competence and, at some level, implicit understanding of how procedures work, as well as the goals, subgoals and constraints of the tasks being taught (Greeno, Riley & Gelman, 1984; Resnick, 1982). This second aspect is considered the knowledge component of expertise (Lesgold, 1983; Glaser, 1983).

Recent work in cognitive psychology has explored the significance of content knowledge in expert performance. (Chi, Glaser, & Rees, 1982; Glaser, 1983; Greeno, 1978; Lesgold, 1983; Voss, Greene, Post, & Penner, 1983). Initial work in the field of expert performance attempted to analyze the structural or procedural aspects of performance devoid of content knowledge, but as research extends deeper into areas that rely on broad substantive knowledge (such as physics, political science, geography), exploration of the content knowledge must be undertaken as well. This trend toward the analysis of domain-specific knowledge has also been useful in the development of expert systems in the realm of artificial intelligence. Some of the more notable cases involve advances in medical diagnosis (Pople, 1981; Shortliffe, 1976), and intelligent computer assisted instruction (Sleeman & Brown, 1983). By doing a detailed analysis of subject matter of fractions on the part of expert teachers, we hope to understand how that knowledge is used in effective teaching.

Relation of Subject Matter Knowledge to Lesson Structure

Arithmetic lessons are not homogeneous, continuous streams of action lasting for forty minutes. Most good lessons contain several segments or structures (Good, Grouws & Ebmeier, 1983; Leinhardt, 1983b). Each of these segments can be analyzed by considering the system of goals and subgoals that mediate the selection of particular actions. These systems of goals and actions can be represented by planning nets (Van Lehn & Brown, 1980). Plans are constructed in response to the need to achieve certain goals (Stefik, 1981; Hayes-Roth & Hayes-Roth, 1979). One of the most salient action segments in the teaching of arithmetic is the presentation of material. Presentations are the activity segment most closely identified with "teaching." Other segments are guided practice, monitored practice, drill, tutoring, etc. It is in the context of presentation that teachers introduce new concepts, present new algorithms, review learned material, and offer explanations. It is also in the context of a presentation that teachers must draw most heavily on their subject matter knowledge.

In order to see how subject matter knowledge is used, brief descriptions of a portion of the theoretical planning net will be provided. Presentation of algorithms is analyzed in terms of four goals: definitions presented, algorithm presented, algorithm learned, and algorithm understood. Figure 1 displays the planning net for the second goal, presenting the algorithm. The planning net contains both the goals (hexagons) and the actions (rectangles) involved in presenting an algorithm. The relationships among actions and goals are captured by labeled links. Consequence links show the actions that, when completed, will achieve a given goal. Goals that are linked to actions by pre-requisite links must be satisfied before that action can be executed. For example, a pre-requisite of demonstrating an algorithm is identification of the steps in that algorithm. Goals that are co-requisites of actions must remain true throughout the execution of the action; for example, maintaining student attention. Planning nets can also include post-requisites, goals that are linked to actions which become true upon the completion of the action, and tests for iterative actions (Greeno, Riley, Gelman, 1984; Newell & Simon, 1973; Sacerdoti, 1977).

In Figure 1, the goal of having the algorithm presented is the consequence of three actions: stating the algorithm, demonstrating the algorithm, and identifying the conditions for use. These three actions require subject matter knowledge for their content selection as well as for the remaining actions and goals. Thus, in order to know which algorithm to state and which demonstration to use, the subject matter knowledge must be activated. As can be seen from the figure, however, the nature of that knowledge remains unspecified. The purpose of presenting the planning net is to contextualize the subject matter knowledge, which is the focus of this paper.

Insert Figure 1 here

Data Source and Analysis

Four expert mathematics teachers and four novices who taught at the fourth grade level were selected.[1] These teachers were a subsample of a set of twelve expert teachers and four novices who participated in a three-year study of expertise (Leinhardt, 1983a, 1983b, 1983c). The expert teachers were selected because of the unusual and consistent growth scores of their students in mathematics over a five year period. The novices were student teachers in their last year of a teacher training program. The subgroup of four expert teachers were chosen from the set of twelve because they taught at similar grade levels. Two of the experts seemed to have high knowledge of subject matter; one had moderate knowledge, and one had low knowledge. The four novices had moderate to low subject matter knowledge. In the first two years of the study extensive data were collected on these teachers: they were observed for approximately 3 months each year; they were videotaped for 10 hours; they were interviewed on several topics, including the taped lessons, planning and evaluating their lessons, and fraction knowledge. They were also given card sort tasks on math topics. Transcription of the resultant protocols became one data base while transcriptions of videotapes and observations of their in-class

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[1]Some of the teachers were male; however, in order to preserve anonymity, we describe all individuals as female.

performance became another.

The analyses were of two types. First, the fraction interview and mathematics card sorts were analyzed to determine any consistent patterns of knowledge and understanding as well as confusion and misunderstanding. Second, three of the teachers, two high knowledge and one middle knowledge teacher, were examined more closely. Videotapes of these three teachers each teaching a lesson on reducing fractions that lasted one or two periods were examined in detail. The teachers taught the lessons in the same progression (spontaneously) and had completed prior lessons in similar sequences. They used the same pages of text and very similar examples. We were trying to establish the differences in content used and communicated by those teachers whose performance was superficially similar, but whose knowledge organization was substantially different.

We analyzed the declarative knowledge base by building semantic network representations of the text material alone and representations for each of the teachers. These semantic nets are quite powerful tools for demonstrating similarities and differences among knowledge bases. The information in a given semantic net was based on videotapes of lessons and the parallel stimulated recalls, with interviews and card sort data used to confirm the presence of a particular concept or relationship. While this type of non-statistical but formal analysis of qualitative data for a small number of cases is new to educational research, it has become a confirmable methodology for psychology (Ericsson & Simon, 1980).

Since the formalism of semantic networks is relatively new to educational research, it will be helpful to discuss their application in somewhat more detail. A semantic network is a node-link structure that contains two types of knowledge: concepts and relations among those concepts. Concepts are represented as nodes while the relationships among those concepts are represented as labels on the links. The information represented in the nodes depends upon the particular domain, and the number of nodes is a function of both the domain and the level of analysis. A relatively simple domain that is analyzed at a high level will have a small number of nodes while a complex domain analyzed at a finer level of detail will have a large number of nodes. The number of links is also a function of the domain and the level of analysis so that a detailed semantic network of a complex domain will tend to have a large number of links. A major constraint placed on the development of a semantic network has to do with the labels and direction of the links. While link labels tend to vary somewhat across domains, there is a fairly well specified set of frequently used labels. One of the most common link labels is *has-prop* which designates one node as being the property of another node. The direction of the arrow specifies which node is the exemplar and which is the property. For example, a node representing the concept **bird** could have *has-prop* links to the concepts of **feathers** and **flies**. (Note, of course, that neither of these concepts are properties of all birds but that they are properties of most birds). Another common label is *is a* which designates a node as being an instantiation of a higher-level concept. For example, the nodes representing concepts for **eagle** and **crow** would have *is a* links to the higher level concept of **bird**. Other common link labels are *has-part*, *subset*, and *is-part*. Several of these link labels describe inverse relationships and it is often the case that only one relationship is shown explicitly.

There are two basic uses of semantic networks in cognitive science research. One common use of semantic networks is the construction of hypothetical knowledge bases used to develop hypotheses about the knowledge that is sufficient to perform a particular task. A second use of semantic networks is to develop models of the problem-solver's knowledge base. In this type of research, the information in the semantic network is based upon the information that is obtained from verbal protocols taken during the problem-solving activity. The use of semantic nets in the current analysis is essentially a combination of these two uses. First, a semantic network was developed that represents the basic fraction knowledge contained in the text. Secondly, semantic networks were developed from transcribed videotapes (and additional material) of the corresponding lessons as presented by different teachers. These semantic networks represent a combination of a core of fraction knowledge plus the information that the teacher discussed explicitly during the lesson presentation.

Results

Overview

The outcome of the card sorts and fraction knowledge interviews confirmed our impression that in spite of high levels of student success for all teachers, two teachers had exceptionally high math knowledge, one had middle-level knowledge, and one had barely sufficient math knowledge. Novices had generally low knowledge, but there were some surprises.

The math sort data indicated that there were natural breaks between those experts with high versus low math knowledge as well as between the experts and novices. Briefly, using both diagramatic trees and sorts we observed the following differences: a) High knowledge experts sorted 45 math topic cards into approximately 10 categories and ordered the topics by difficulty to teach or perform. They also grouped addition and subtraction together and then ordered problems through to decimals. b) Novices made categories for every one or two problems and noted little differentiation in difficulty. They also indicated almost no internal connections.

With respect to teaching fractions, similar distinctions appeared in the tapes and interviews. For example, although all teachers taught equivalent fractions, when queried about the equivalence of 3/7 to 243/567, the less knowledgeable teachers tended to get 81 as a factor and then to say either that the fractions were not equivalent or that they did not know. Also the less knowledgeable individuals, when discussing equivalent fractions, did not mention that to raise or lower a fraction you multiply or divide by a fractional equivalent of one (in this case 81/81), nor when interviewed did they seem to realize it was true. In contrast, the two teachers who had greater math skills immediately saw the equivalence and reported it. Both of these teachers when they were teaching noted the fact that equivalence occurred <u>because</u> the original fraction was being multiplied by one.

Interviews

Table 1 summarizes twelve items from the fraction interview. Most of the teachers and novices tended to answer a subset of eight items in a reasonable way, with some exceptions as noted in the table (darkened lines point out discrepancies). Four of the items seemed to discriminate between groups of teachers in an interesting way. In defining a fraction, seven of the teachers referred to equal parts of a whole, thus retaining the notion of equal segments and their relationship to the whole. One teacher defined a fraction as the points between zero and one or zero and any other whole number, including the whole numbers. This teacher was the only one who consistently used the number line as a frame for the lessons and the only one who saw fractions as having a measurement property.

Perhaps the most telling item was one which followed an item that asked for a definition of equivalent. All teachers defined equivalent correctly, emphasizing the regional equality. The next question was "are 3/7 and 243/567 equivalent?" The example is, in some sense, "illegal" because it takes a simple construct and pushes it out of the normally observed range. The two high and one middle knowledge expert saw very quickly that 81 was a common factor and that the fractions were therefore equivalent. That is, they found 81 and recognized its significance. Our lower knowledge expert and two novices found 81; but they did not know what to do with it and eventually said that the fractions were not equivalent. As Ms. Lawn said,

> "No, wait, let's see. Well, I'm saying that because you can divide 3 into 253 and 7 into 567, ahm, huh, not necessarily because you cannot, as they are, 3 and 7 don't both go into these numbers evenly (that is that 7 doesn't go into 243) . . . Okay, no, I'm just figuring, it's 81 over 81, if you divide it out that way . . . Isn't that funny, I teach, I'm teaching that and understanding it when I'm teaching it. Yeah it does, no, they aren't equivalent because you can't divide them by the same number. Like, I would have to divide 567 in order to see if they were equivalent and it isn't. Let's see 21 . . . no, they aren't. Can you tell me?"

This level of confusion was present for two other teachers, one expert and one novice. One other novice refused the item, and one (Ms. Benny) got it by reasoning, "243 divided by 3 equals 81 times seven equals 567," but she did not explicitly mention that 81 was common a factor. Thus, among the teachers there is a clear break in understanding, where the rule can be successfully stated by some but not applied by them in extreme cases.

A third item of interest involved the concept of unit. The teachers were asked to draw pictures representing 3/4, 5/5, and 5/4, which all but one did successfully. They were also asked to indicate the unit for each of these (4/4, 5/5, 4/4, respectively). Everyone correctly answered the first two, but our high and middle knowledge experts and one novice (Ms. Benny) were the only ones who recognized one as the unit base of 5/4; two low knowledge teachers said that the five segments (numerator) were the unit; and one teacher, our low knowledge expert, said that the unit was two wholes. The concept of unit is important to fractions because it allows one to move back and forth from discrete to continuous models without losing the important relations. For example, 5/8 is not equivalent to 5/4, but the same number of pieces are involved in both numerators and denominators if one is referring to two rectangles each divided into four segments. If the unit is blurred from one to two, the fraction meaning can be as well. One of our novices first said that the five-fourths item was impossible to draw. She then drew five circles, divided each one into fourths, and shaded one-fourth of each of the five circles (5/20). This represents direct verbal translation of the problem from five-fourths to five one-fourths.

The fourth item that produced unexpected results was designed to determine the teachers' understanding of ratio in relation to fractions. We chose the ratio problem for two reasons: a) ratio is taught in fifth grade and these were fourth grade teachers who might have students ready to move ahead; and b) ratio uses the notation of fractions but not the operations. The item first asked how ratios were similar to or different from fractions. Then, the teachers were shown a figure and asked to specify what fraction was shaded, and the ratio of unshaded to shaded. Essentially, ratio represents the numeric relation between two things (part/whole, part/part, speed/distance, etc.). Ratios are not numbers on the number line and can undergo mathematical operations only under particular circumstances. However, ratios can be expressed using fraction-like notations and can include either the part/whole concepts in fractions or the part/part concepts. Not one of our teachers indicated that they knew any of this. All either said that a fraction and ratio were identical or similar, or said that they did not know. One novice, Ms. Mark, said ratios showed relations. When shown an item that had dots arranged in a ratio of 2 to 4, she expressed the ratio correctly. However, when asked for an equivalent ratio, she could only repeat her 1/3 fraction answer. One teacher gave an incorrect response of 2 to 1; the other six were able to state the ratio as 2 to 4.

Presented Lessons

Given the apparent disparity between the ability to express an algorithm and the knowledge underlying that algorithm (understanding), we decided to focus on the representation of subject matter knowledge as presented in classroom lessons by three experts who had adequate subject matter knowledge. We selected the one or two lesson sequences on reducing fractions. The lessons were examined by constructing semantic nets that describe the content of the subject matter presentation. In the following discussion, we present the core material shared by all presentations; then we examine the textbook presentation; and finally we present an analysis of the three expert teachers. In describing the teachers, we start with a brief summary of the lesson flow and then present the semantic net for the subject matter presentation.

Core. Figure 2 displays the semantic net for the core of knowledge involved in reducing fractions. By "core" we mean the concepts that are shared among the three expert teachers and the text. This core is based on the concept **fraction**. A **fraction** *is a* **number** and *has* the *parts* **numerator** and **denominator**. A **fraction** also *has* the *property* that it can have different **representations.** By **representations** we mean that a fraction can be represented or modeled in a non-numeric system. One of the critical concepts of the core is that **equivalence** *is a* **relation** between two fractions. A judgement of **equivalence** can be made when each of two **fractions** *is an* **input** to that relation. The concepts represented in this core or nucleus can be connected to other concepts and it is these additional concepts and relations that distinguish the teachers from each other as well as from the text.

Insert Figure 2 here - Core

Text. In addition to the core of fraction information, the text includes information about alternate representations for fractions as well as information about the relationships between fractions and the operations of multiplication and division.

Insert Figure 3 here - Text

The semantic network shown in Figure 3 describes the information contained in the text and includes three alternate representations for fractions. The most common of these representations involves the concept of a **region** and appears in the lower right side of the network. The text explicitly states that a regional representation *has* the *property* of **shape**, which can be among other things a **rectangle** or a **circle**. The critical relationship that is not explicitly discussed in the text is the fact that a **region** *is a* **unit whole** (shown by a dotted line). The other concepts and relationships shown for the **region** representation are explicitly stated in the text. The basic notion represented in the upper portion of the **region** representation is that a **unit whole** can be divided into two **parts**, one **shaded** and the other **unshaded**. Through a process of mapping, the number of **units** represented in the **shaded part** of the **region** is shown to correspond to the **numerator** of a **fraction** while the total number of **units** corresponds to the **denominator** of a **fraction**. As long as the size of the units is kept equal, units can be added (by drawing lines) or removed (by **erasing** lines) and the **fractions** represented by the number of **shaded** units to the total number of units will remain **equivalent**.

An alternate form of **representation** discussed in the text is the **number-line**. The **number-line** and its relationship to equivalent fractions is not covered extensively in the text. Most of the information regarding **number-line representations** is provided by a single diagram with very little supporting discussion. This minimal coverage is evidenced by the small number of nodes connected to the concept of the **number-line**. The primary relationships are that a **number-line** *has* the *property* of **points** that are **labeled** by **fractions**. The text also introduces the notion that there can be several different **number-lines**, each of which *has* the *property* of a **family**. The fact that this particular *has-prop* link is dashed represents the fact that the text did not explicitly mention the concept of a **family**. This concept was adopted from one of the teachers who explicitly uses the term (see discussion of Konrad for details). The expression of **equivalence** is accomplished by "lining up" a group of **number-lines** in a vertical array. If the **number-lines** have equal sized units, it is possible to show how 1/2 on the "2" **number-line** is **equivalent** to (lines up with) 2/4 on the "4's" **number-line**.

The third form of **representation** discussed in the text is **discrete objects**. Like the concept of a **region**, the **discrete objects representation** is implicitly linked to the concept of the a unit whole (**unit set-1**). This set is further divided into two *subsets* of objects, **unit set-2** and **unit set-3**. The members of **unit set-2** are defined as having a particular property, **prop-1** (e.g., bottles). The members of **unit set-3** also have this property but also have an additional distinguishing property represented by **prop-2** (e.g., full of liquid). In constructing the value of a fraction, the quantity associated with the number of members of **unit set-3** corresponds to the **numerator** and the quantity associated with the number of members of **unit set-1** corresponds to the **denominator**. The connection between discrete objects and the unit sets is not explicitly discussed in text. Unlike its discussion of **region representation**, the text does not provide description of how the **discrete object representation** can be used to discuss **equivalent fractions**. Rather, the **discrete objects representation** is used only to show that the fractional part of **unit set-1** *has-prop* **prop-2** (e.g., 3/6 of the bottles are full).

In addition to the alternate **representations**, the text provides general descriptions of how the **multiplication** and **division operators** can be used to generate **equivalent fractions**. The basic concepts associated with these **operators** are **reducing** to **lower terms**, and **equivalence.** As shown in the upper portion of Figure 3, there is a relatively close relationship among the multiplication and division operators and the concepts of reducing and equivalence. The main points expressed in the text are that **equivalent fractions** can be generated by either **multiplying** or **dividing** the **numerator** and **denominator** of an existing **fraction** by the same number. This is represented by the the relationships among the *inputs* and *output* of the the **operators** and the **equivalence relation**. Basically, both the **numerator** and the **denominator** of a **fraction A** can be multiplied or divided by **B** and the result will be an equivalent fraction **A'**. **Division** is explicitly mentioned to *have* the *property* of **reducing** and hence creating a **fraction** in **lower terms**. Furthermore, **division** *has* the *property* of being **restricted** in that both the **numerator** and **denominator** of **A** must be evenly divisible by **B. Multiplication**, however, has no restrictions on the numbers that may be used in the operation.

The text's description of the general concepts and operations involved in equivalent fractions includes most of the critical relationships. What the text fails to provide is any explicit description of how these concepts and operations are to be applied in a problem solving situation. The only way to acquire this knowledge from the book, would be through the process of induction in which the student acquires an understanding of the proper rules and concepts while working through the problems. Although induction is a powerful learning mechanism that can lead to the acquisition of correct concepts and rules, it is also a relatively inefficient use of cognitive resources and can result in the acquisition of misconceptions and incorrect rules.

The omission of this critical information from the text was not an oversight but was an intentional decision. Rather than providing the students with a lengthy discussion of the various algorithms and heuristics that can be applied when generating equivalent fractions, the text is designed to provide a general framework that can then be elaborated on by individual teachers. In the course of the lesson, teachers can introduce whatever algorithms and heuristics they feel are appropriate. In the following sections, we discuss how different teachers handle the task of teaching the concepts and operations for reducing fractions.

Konrad. Ms. Konrad presented her lesson during an eleven-minute presentation on day one and a seven-minute follow-up on day two. The presentation (which was followed by guided practice and individual practice) contained six segments. First Konrad reviewed the fact that A/A equaled one and that any fraction in which the numerator and denominator were the same equaled one. Secondly, she reviewed the fact that any number (including a fraction) when multiplied by one yields that same number (or fraction). Konrad connected segments one and two by noting that such multiplication produces equivalent fractions.

The third segment was the beginning of the new material. Konrad noted that whole numbers divided by one yielded the same number and then said fractions divided by one or a fractional name for one returned the same number or fraction. She demonstrated this by dividing 3/6 by 3/3 and noted that the answer yielded an equivalent fraction pair, 1/2 and 3/6. This was supported by a pictorial representation. Two equal-sized rectangles were drawn, one divided and shaded to show 3/6, the other to show one-half; and students were invited to compare them. This was a minor mistake in that the idea of the operation would be more easily conveyed with one drawing rather than two.

The fourth segment was complex. In this segment Konrad showed that while one can multiply by any fractional name for one (any of the numberline families) and get an equivalent fraction, dividing by fractional names for one is restricted. The fifth segment continued the divisional discussion and presented the notion of iterative division as a precursor to lowest terms. The final segment labeled the process--reducing to lowest terms. The second lesson reviewed examples and added hints for knowing when reducing had been completed.

Insert Figure 4 here - Konrad - generic

The semantic network shown in Figure 4 was developed from the information contained in Konrad's class presentation and interviews. Konrad uses two of the alternate **representations** presented in the text, the **region** and the **number-line**. The **region representation** shown on the right side of Figure 4 differs from the text in that Konrad uses two **unit wholes** rather than one. Each **unit whole**, however, is essentially the same as the one described in the text. The demonstration provided in the text showed that as long as the parts of a **unit whole** remained equal in size, the addition or deletion of interior lines would not affect the overall relationship between the **shaded** part and the **unit whole**. Konrad, on the other hand, uses two entirely separate **region representations** (rectangles) to demonstrate the same basic point. Given two rectangles of equal size with equally sized **shaded subparts**, the addition or deletion of interior lines to either **rectangle** produces a different fractional representation, but does not change the **equivalence** of those **fractions**. However, as shown in Figure 4, the student has to make the inference of equivalence of area between the two rectangles.

The other **representation** used by Konrad is the **number-line**. Here too, the network depicting this concept differs from the text. Konrad goes to great lengths to show how a set of **number-line families**, each defined according to the **value** of a **whole number**, can be used to represent the **equivalence** of two **fractions**. Konrad's operation for achieving this is similar to that described in the text. That is, vertically aligned **number-lines** can be used to show how the **fraction** 2/4 in the "4" **family** is **equivalent** to 3/6 in the "6" **family**. In addition to elaborating on the description offered by the text, Konrad uses the set of **number-lines** to introduce the concepts of **proper** and **improper** fractions and the concept of the **identity element** or the "fractional name for one" that exists in each **number-line family**.

The **identity element** is the key to Konrad's method of teaching the concept of **equivalence**. We elaborate this in Figure 4 which shows the basic information that Konrad uses to describe the relationship between the **identity element** and **equivalent fractions**. The primary things to focus on are the relationships among the *inputs* to **multiplication** and the concepts of the **identity element** and **fraction**. These relationships show that **multiplication** of a **fraction** by the **identity element** will always yield an **equivalent fraction** regardless of the value of the **fraction** or the value of **numerator** and **denominator** of the **identity element**. For example, 2/4 multiplied by 4/4, 5/5, or 300/300 will each yield an equivalent fraction.

Insert Figure 5 here - Konrad multiplication/equivalence

Figure 5 shows the another step in Konrad's lesson, that is, the **division** of a **whole number** by the **identity element**. The primary points of focus in this figure are the relationships among **division** and its *inputs*. One of the *inputs* to **division** is implicitly specified as the **dividend** while the other *input*, the **identity element** is implicitly specified as the **divisor**. These relationships are implicit because Konrad does not use these terms but emphasizes the point that **A** is <u>divided</u> <u>by</u> **B**. Notice that the *inputs* to

equivalence are **numbers** but they are not identified as **fractions**. Instead, **A** is defined as a **whole number** which *satisfies* the *restriction* that is placed on the **division** of **fractions**.

Insert Figure 6 here - Konrad division/equivalence

Figure 6 shows the final step of Konrad's lesson in which **division** by the **identity element** is applied to a **fraction**. This is represented by the reintroduction of the *is a* relationship between **number** and **fraction**. Recall that this link is absent from the prior Figures 4 and 5. Once this relationship is restored, however, there is a change in the way in which the **restriction** on **division** can be *satisfied*. In order to *satisfy* this **restriction** both the **numerator** and **denominator** of **A** must be **evenly divisible by the numbers** in **B**. If this constraint is met, then **division** *has* the *property* of **reducing** the original **fraction** to **lower terms**.

Insert Figure 7 here - Konrad division/reducing

It should be apparent from the number and density of the semantic networks describing Konrad's lesson that she is providing the students with a rich body of conceptual information. On the other hand, at no point in time does she provide the students with an explicit description of an algorithm for reducing fractions to lower terms. On the surface, it would appear that Konrad's lesson places a demand upon the students to induce the proper rules that will lead to problem solutions. In some ways this is true, but it is important to note that, as opposed to the text, much of the information in Konrad's lesson can be viewed as providing implicit rules for applying the multiplication and division operators. In the case of multiplication, for example, the use of the identity element will always lead to an equivalent fraction. What is missing is a discussion of how to select the particular identity element that will lead to the desired fraction. In the case of division, Konrad uses the implicit notion of restrictions to guide the selection of a proper identity element. Adherence to these restrictions will lead to an equivalent fraction, but again not necessarily the desired one. In the subsequent lesson, Konrad offers a number of "hints" about the choice of identity elements as well as some "hints" about when a fraction cannot be further reduced.

These hints take the form of heuristics that guide the process of reducing fractions to lower terms. A heuristic can be thought of as a "rule of thumb" that will usually but not always lead to a problem solution. An algorithm, on the other hand, defines a specific set of operations that, if correctly applied, will always lead to a solution. The obvious difficulty with algorithms is that they are of little use in situations where they cannot be applied. While heuristics may not always lead to a correct solution, they can be applied to a broader range of problems. It is not clear when or how these two problem solving methods should be taught, but it is clear that the process of teaching heuristics is different than that for teaching algorithms. This difference can be seen by comparing Konrad's lesson with the lessons of the following two teachers who teach explicit algorithms.

Wall. Ms. Wall gave her lesson during a single eight-minute presentation. Like Konrad, she also had six segments in her presentation. However, she started with the, segment that had ended Konrad's lesson--namely, labeling the process as reducing fractions to lower terms. In the second segment, Wall drew a rectangle, marked off 3/6 and erased the thirds lines to "reduce" it to 1/2. She then asked the class how 3/6 got to 1/2, eliciting the third lesson segment--the operation of division. The fourth segment skipped to the notion of equivalent(ing) fractions by multiplication and its opposite, reducing fractions by division. The fifth segment was the crux of the lesson. The algorithm was presented as follows: a) determine whether the numerator is a factor of the denominator; if so, divide and stop; b) if not, list the numerator's factors, starting with the largest and test until a common factor is found for both numerator and denominator and divide by it. Finally, Wall worked through several problems. The presentation was very clean and sparse; responses from students were those the lesson required, and the algorithm got taught. Little time was devoted to conceptual development.

Figure 8 shows a semantic network representing the conceptual information that Wall includes in her lesson on equivalent fractions. The "core" information is essentially the same as Konrad's, but there is no link showing that **equivalence** *is a* **relation**. Another thing to note is that **reducing** and **equivalence** are represented as the *output* of **division** and **multiplication**, respectively. In addition, these two concepts are treated as opposites, in that **reducing** *has* the *property* of **making smaller** while **equivalence** *has* the *property* of **making larger**.

Insert Figure 8 here - Wall - generic

The other major part of this semantic network is the **region representation** shown on the right hand side of Figure 8. This portion of the structure is similar to that provided in the text. The primary difference is that Wall uses the operation of **erasing** lines to represent the process of **reducing** rather than the relationship of **equivalence.**

The major part of Wall's lesson is captured in her very detailed presentation of an algorithm for reducing fractions. This procedure is explicitly taught and is very efficient. Figure 9 shows the procedure in a flow chart which reads from top to bottom. Given the objective of reducing a fraction, examine the numerator and see if it is a factor of the denominator; if it is, divide both the numerator and the denominator by the numerator and the new fraction is in lowest terms. If the numerator is not a factor of the denominator, then list all factors of the numerator, select the largest and test to see if it is also a factor of the denominator. Keep going down the list of factors and when the largest common one is found, divide by it; then the fraction will be in lowest terms. In contrast to Konrad's procedure, this is non-iterative and requires no checks for reducibility.

Insert Figure 9 here - Wall - algorithm

There are some apparent differences between Wall's lesson and the information provided by the book. Wall focuses on a single form of representation and does not attempt to provide the students with a detailed understanding of the various concepts surrounding equivalent fractions. Rather, she provides a relatively sparse conceptual framework and a very efficient algorithm for reducing fractions to lowest terms. Wall's lesson also involves some erroneous information that could eventually cause confusion on the part of the students. The most obvious of these errors is the relationship between reducing fractions and "making smaller" or "taking away."

Yoda. Ms. Yoda gave two lessons on reducing fractions--one lasted two and one half minutes; the other, ten and one-half. The combined presentations consisted of a total of seven segments. She started by centering the notion of equivalent fractions, saying the class had been using multiplication and now they would use division. The second segment reviewed the "cloud" notion from the text, and the lesson ended. The next day, the first segment (third in sequence) reviewed the labels of reducing and equivalence. The next two segments were discussions of odd and even numbers. In particular, she reviewed the fact that even numbers have the property of always being reducible by two and possibly iteratively. Yoda's sixth segment introduced the label of lowest terms and the idea of needing to find the largest common factor. When that factor is found (just how is not specified), division by it produces a fraction in lowest terms. Yoda then worked several examples without further explanation.

Figure 10 shows a semantic net representing the conceptual information in Yoda's lesson plan. As with the previous semantic nets, Yoda's lesson involves a relatively explicit description of the "core" information. In addition to this, she discusses the notion that **numbers** can *have* the *properties* **even** or **odd** and that fractions can be made **larger** via **multiplication** and **smaller** via **division**. Like Wall, Yoda discusses the fact that **reducing** *has* the *property* of **making smaller**. The concept of **making larger**, however, is only seen as a property of **multiplication** and there is no direct link between **making larger** and **equivalence**. The majority of Yoda's lesson involves the description of an algorithm that the students can use to reduce fractions. This algorithm is shown in Figure 11.

Insert Figure 10 here - Yoda - generic

The algorithm taught by Yoda is neither as succinct, nor as reliable as that taught by Wall. In addition, during the course of teaching the algorithm, Yoda makes several critical errors. In teaching her reducing algorithm, Yoda attempts to teach a simple reduction method and then build on this until she presents a complete algorithm. She begins with an algorithm that is relatively simple but applies only to certain types of problems.

Insert Figure 11 here - Algorithm 1

Figure 11 shows the basic algorithm Yoda presents. The initial steps involve writing the fraction, putting division signs next to the numerator and denominator, putting "little clouds" next to the division signs, and writing an equal sign to the right of the "little clouds." These initial steps are all designed to provide a format for working on the problem. The "little clouds" are used by Yoda to designate where the reducing numbers are to be written and to prevent the child from losing track of the original problem. The first critical step involves testing whether or not the numerator and denominator are <u>even</u>. This test is not valid for all reduction problems but since the first example Yoda chooses involves even numbers in both the numerator and denominator she is able to proceed with the problem. The next steps involve writing a "2" in each of the little clouds, dividing the numerator and denominator by "2" and then writing the results to the right of the equal sign. In order to determine if the original fraction has been reduced to lowest terms, it is necessary to test the result of the division. At this point in her presentation, Yoda changes the test from determining if the numerator and denominator are even, to determining if there is a number that will go evenly into both. This is shown in the flow chart by the dashed line connecting to two test diamonds. If there is no number that will go evenly into the numerator and denominator the fraction has been reduced to lowest terms and the procedure is finished. Since the example Yoda chose was in lowest terms following the first division, there was no opportunity to discuss what another divisor would be.

At this point, Yoda selected a new problem that was considerably more difficult than the first. In fact, this problem could not be solved by following the steps in Figure 11. The steps to solve this new problem are shown in Figure 12.

Insert Figure 12 here - Algorithm 2

The first step involves "finding the biggest number that goes into the numerator and denominator." Presumably, Yoda intended this step to be a generalization of the test shown in the initial flow chart. Unfortunately, there was no explicit mention made about the relationship between the first and second problems. The next step shown in Figure 12 is "think what two numbers multiplied equal the numerator." The essential aspect of this step is factoring the numerator but again there is no explicit mention of this process. One selects the largest number that goes into both numerator and denominator evenly and writes it in the "little clouds." Then that number is divided into both the numerator and denominator, and the answer is written to the right of the equal sign. The next step is to determine if the fraction has been reduced to lowest terms. This is accomplished by seeing if the fraction is even or odd. If the fraction is even then it is necessary to repeat the process. If the fraction is odd, an additional step is involved to determine if there is a number that goes evenly into both the numerator and denominator. If such a number exists, then it is necessary to repeat. If no such number exists, the fraction is in lowest terms and the procedure stops. One additional thing to note about this algorithm is that Yoda simply states that if the fraction is not in lowest terms, it is necessary to "go further" without specifying exactly what that entails. Yoda's lesson lacks both a detailed conceptual framework and an efficient algorithm. Like Wall, Yoda's presentation of equivalent fractions involves a misconception about the relationship between reducing and "making smaller." However, both Yoda and Wall support the demanding (demanding in the sense of cognitive load) algorithmic string with effective use of external memory devices--in Yoda's case "the clouds" and in Wall's, the bracketed factor list.

Summary and Conclusions

Summary.

We examined a group of expert teachers and novices. Expert teachers had shown similarly high levels of student growth over time while novices were beginning teachers. Among this set we noticed considerable variability in their knowledge of fundamental fractions concepts. The differences between novices and experts were consistent with other research on expertise (Chi, Glaser & Reese, 1982; Larkin, 1983) namely, experts had more elaborate and deeper categories while novices had more horizontal separate category systems for problems. However, among the experts there were differences in levels of subject matter knowledge. We selected three experts who all seemed quite similar in their knowledge of fractions and in their lesson coverage. In-depth analysis of the explanation behavior, however, revealed substantial differences in the details of their presentations to students. Specifically, there was considerable difference in the level of conceptual information presented as well as differences in the degree to which procedural algorithmic information was presented. Secondly, teachers had decidely different emphases in their presentations--they entered the topics differently. Konrad approached the topic of reducing fractions through the identity element, while Yoda and Wall approached it via the contrast with finding equivalent fractions. Finally, we noticed differential representation systems: numberline, regional, and numerical.

There are several general issues that emerge from these analyses with respect to student learning and teacher competence. One issue is the fact that textbooks and teachers often provide incomplete descriptions of the concepts and relationships in a domain. In general, the less complete the student's knowledge base, the greater the likelihood that the student will generate incorrect inferences, develop misconceptions, and produce inaccurate problem situations (Resnick, 1980). Specifically, with respect to equivalence, equivalence can be maintained by either reducing or raising a fraction. The book and the teachers, however, focus on the maintenance of equivalence by raising a fraction by multiplication and fail to note the symmetry of multiplication and division. In no case was the interrelationship of these concepts made explicit. In fact, one teacher explicitly mentioned that equivalence and reducing are opposites.

A second related area has to do with the mapping between the numerical representation of fractions and the alternative representations. A striking example involves the mapping between equivalence of numbers and regions. The fraction, 2/4, is often represented by a rectangle divided into four parts with two of those parts shaded. In order to show how 2/4 is equivalent to 4/8 with numbers, 2/4 is multiplied by 2/2. The comparable action in the rectangle is to bisect the existent figure with a single line to create eight total parts, four of which are now shaded. In order to understand this comparison, several intuitive points must be considered. First, multiplication by 2/2 is equivalent to drawing a single line. Secondly, and potentially more confusing, drawing a line is actually a division process and yet this is somehow equivalent to multiplication by 2/2. Mathematically, these relationships are quite precise, but on the rational level, the mapping is not at all straightforward. An additional problem arises in children's conceptions of fractions: they often confuse the concepts of larger and smaller when dealing with fractions. For example, some children judge 1/8 to be larger than 1/4 because 8 is larger than 4. This type of misconception can further hinder an understanding of equivalence.

A major objective in current efforts to improve mathematics competency in children is to improve the reasoning and understanding of conceptual aspects as opposed to simple skill development. Until recently, however, we have had no systematic way to identify the components of such competency. The detailed analysis of systems of knowledge hold promise for such identification. Components of competency involve multiple representations, understanding the function of basic arithmetic principles, such as the identity function, and multiple linkages across concepts that are used in any one aspect of arithmetic. Semantic networks permit the display of these elements and their links.

With the ability to display the system of knowledge, we can determine both if teachers appear to have conceptual understanding themselves and if they transmit it in their explanations. If they do not, it is possible to remedy the situation in a straightforward fashion. Namely, we can construct in-service support that is tied to lesson presentation rather than independent thematic issues; we can expand the conceptual and linkage information in marginal notes of teacher's manuals; and finally, we can make use of pressing examples (such as 3/7 and 243/567) to expand the application of principles. As teachers increase their conceptual knowledge and become more fluid in connecting their knowledge to lesson presentations, their students' mathematical competence should also improve.

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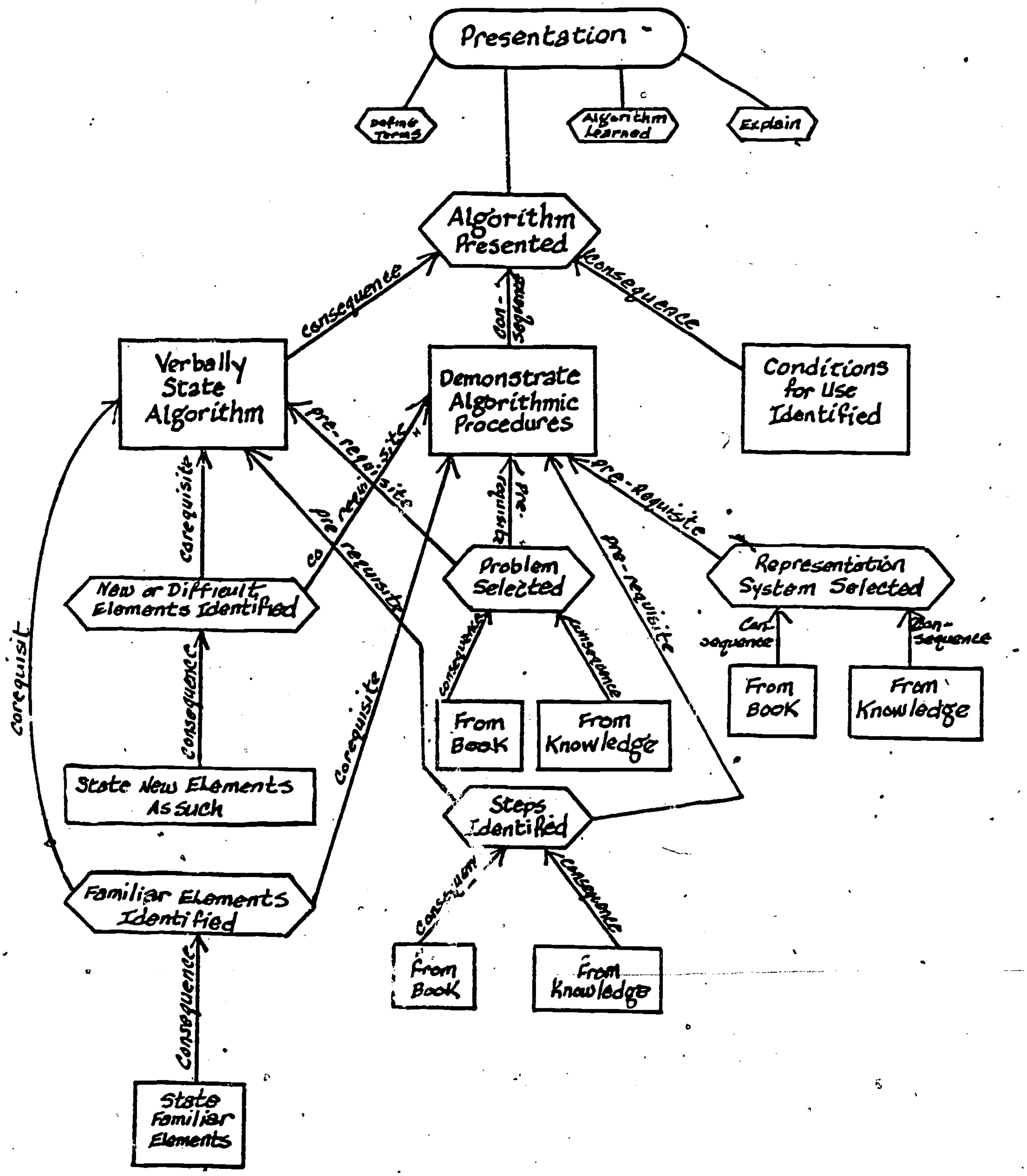
Table 1
Sample of Responses to Fraction Interview

	Experts				Novices			
	Konrad	Yoda	Wall	Rivers	Lawn	Spark	Benny	Mark
Define a fraction	Segment between 0 - 1 or other whole numbers	Equal division of something	A part of a whole	An equal part of a whole	A part of a whole	A part of a whole	A piece of a whole	Parts of a whole
What do students have trouble recognizing as fractions	Mixed numbers	They don't	0/4, 9/5	3/1	1/2 because it's too familiar	Common denominators [sic]	Number and fraction [mixed number]	Fractions beyond one whole 5 1/5
What does "a reduced fraction" mean?	Dividing by one renamed as a fraction	Making it smaller Dividing through by the same number	Bringing it to lowest terms	No number can go in evenly	Use a common denominator and take to smallest number	Dividing the numerator and denominator by the largest number	Broken down into smaller parts than any other	Lowest terms
How do you know it's reduced?	No numbers left to divide by	Can't divide	Can't divide numerator or denominator	Two numbers not related	Can't reduce further	Can't divide	Vampires who hate to see a fraction not reduced	No longer divide it
How do you know if 2 fractions are equivalent?	2 fractions same amount	Same area; could multiply by number (1) to get another	Equal; could multiply or divide to get the other	"Two equal not in size but number" No - the opposite	Equal to same amount	Multiply or divide by same number	2 fractions mean the same thing	1/2 = 3/6: "equivalent fractions are halves of a whole"
$\frac{3}{7} \overset{?}{=} \frac{243}{567}$ equivalent?	Yes; 81	Yes; 81 goes into both	Yes; divide by 81 get 3, 7	Yes; 567 ÷ 7 = 81. No, 81 is a whole number Not equivalent.	Both 3, 7 get 81 but both numbers don't go in — no.	Tough — No, shut off recorder. 81 is for both. No.	243 ÷ 3 = 81 x 7 = 567. Yes	Can't do it. Don't know.
Define proper/improper fraction	Less than or more than 1	Numerator smaller or bigger than denominator	Not sure numerator smaller; numerator bigger	Numerator bigger	Numerator smaller or bigger	Numerator smaller; more than 1	Don't know. Don't know	Top smaller/larger than bottom
Models for fraction. 1st choice – others	Number line, figures. Paper fold is tough	Area, drawing	Area, not discrete Hard to draw number line	Discrete — because of sets, area. Love number line but don't have one	Objects. Number line not concrete	Region — more visual	Regions. Don't understand number line	Number line → Can manipulate sets, too
Draw and tell unit for 3/4, 5/5, 5/4	OK OK OK	OK OK OK	OK OK Unit: all 5 pieces and 1	OK OK Unit: 2	OK OK Unit: 5 segments	OK OK Prompt but probably OK	OK OK Unit one wreath	→ OK OK Unit is 5 wholes
Example of fraction of whole	1/4 of 16	2/3 of 12	2/3 of 18	Can't generate	With prompt 2/3 of 18	40 out of 100 m&ms or 35 out of 100	5 bunnies 2 of 5 bunnies	2/5 of dominoes with 2 dots
What is a ratio and how is it like a fraction?	Same as fraction	Ratio is a fraction	Don't know	No difference	Not asked	Not asked	Don't know	One figure compared to another
Ratio of unshaded to shaded	2 : 1	2 : 4	2 : 4	Can't be done	1 : 2	2 : 4	1 : 2	2 : 4
Why does it work to multiply by the same number	Multiply by 1	Multiply by 1	Multiply top and bottom by same equals 1	Multiply by same number	Multiply by same number	Because it's the same number	Because it's the same number	Because it's the same number



Presentation
Define Terms
Algorithm Learned
Explain
Algorithm Presented
consequence
Con-sequence
consequence
Verbally State Algorithm
Demonstrate Algorithmic Procedures
Conditions for Use Identified
pre-requisite
pre-requisite
co-requisite
corequisite
pre-requisite
Pre-requisite
pre-Requisite
Problem Selected
Representation System Selected
New or Difficult Elements Identified
consequence
From Book
From Knowledge
Con-sequence
Con-sequence
From Book
From Knowledge
corequisit
Consequence
Corequisite
State New Elements As Such
Steps Identified
Familiar Elements Identified
consequence
consequence
From Book
From Knowledge
Consequence
State Familiar Elements

Figure 1

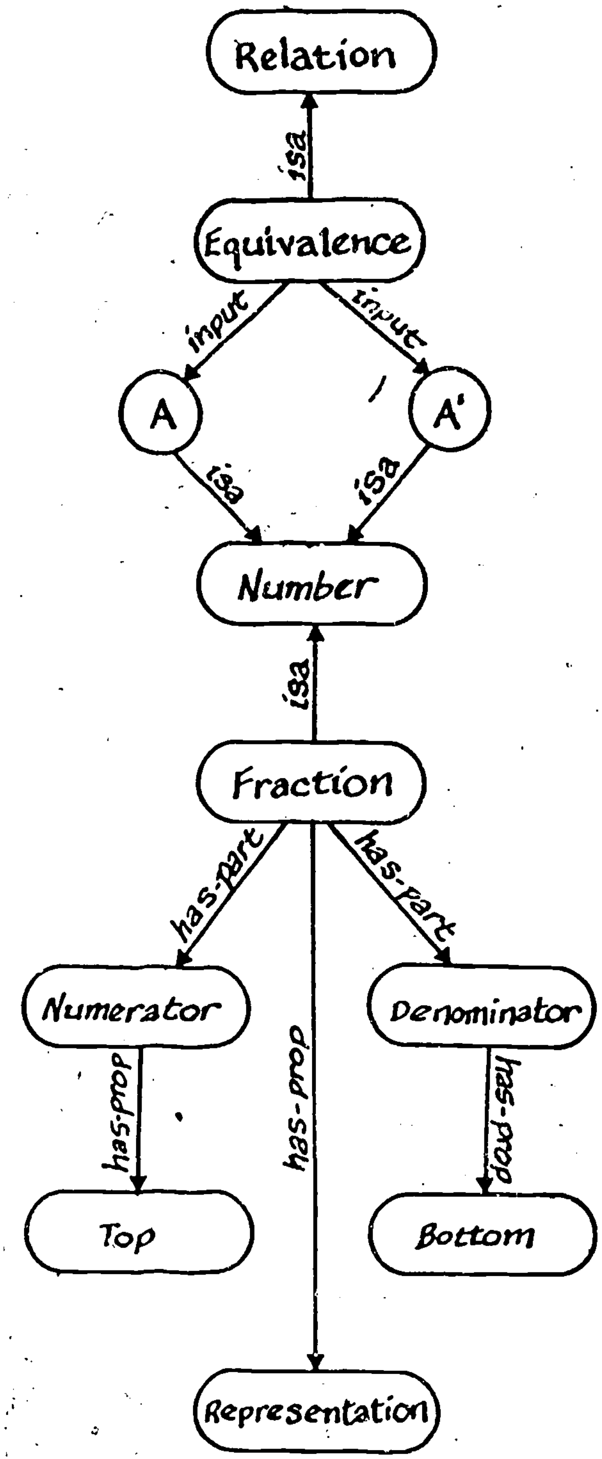


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Figure 2: Core

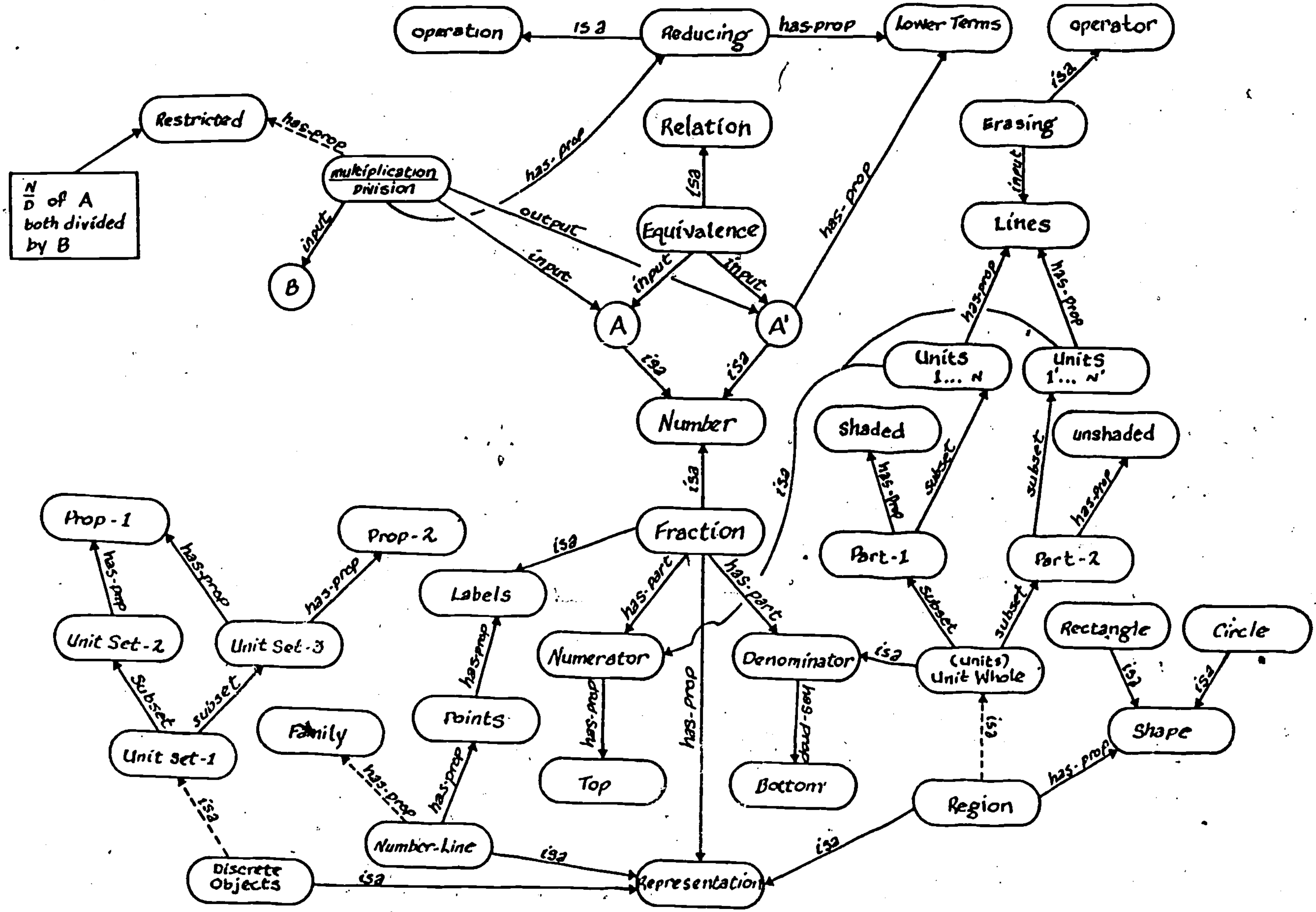


Figure 3: Text

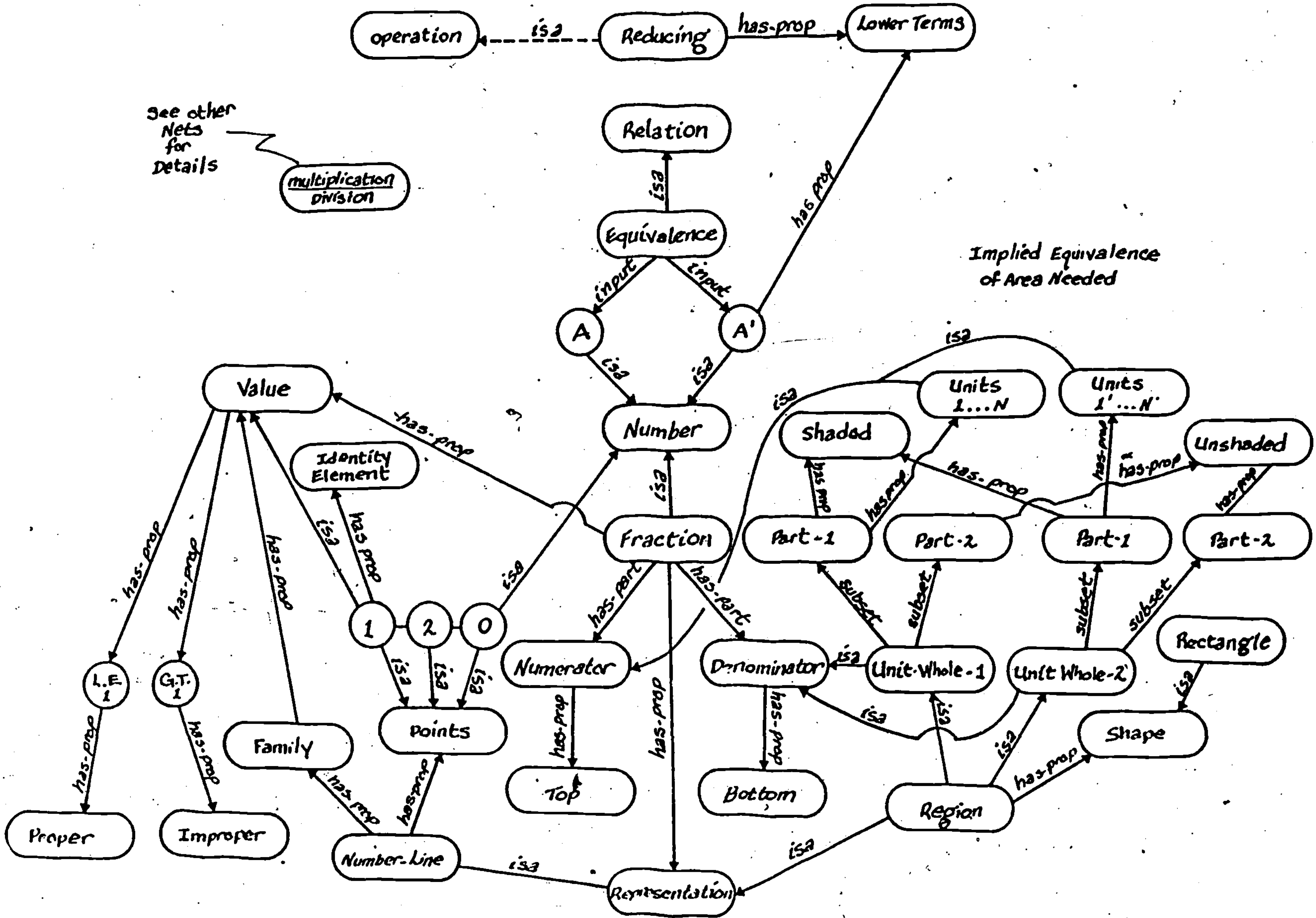


Figure 4: Generic Konrad

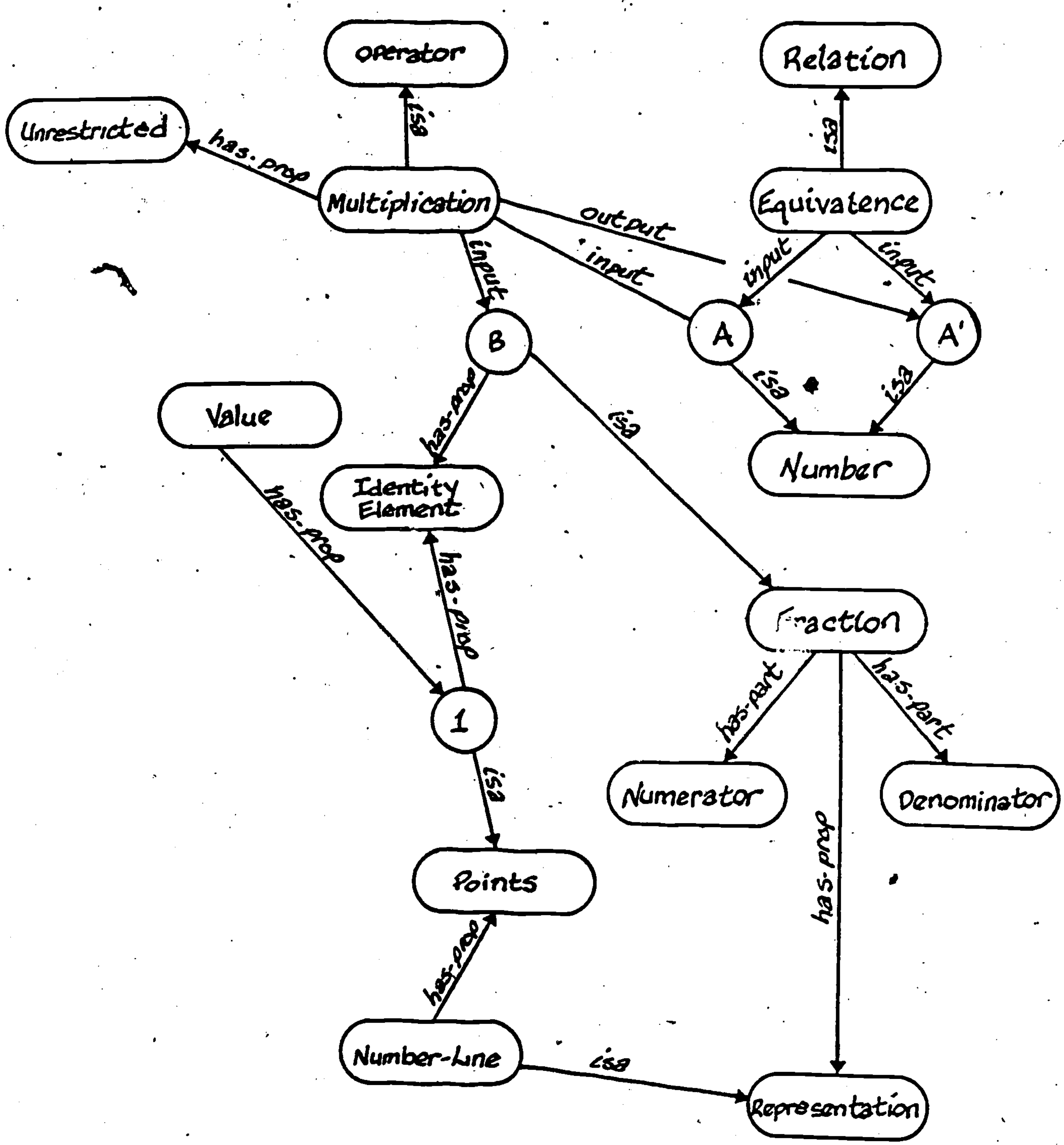


Figure 5: Konrad—Multiplication/Equivalence

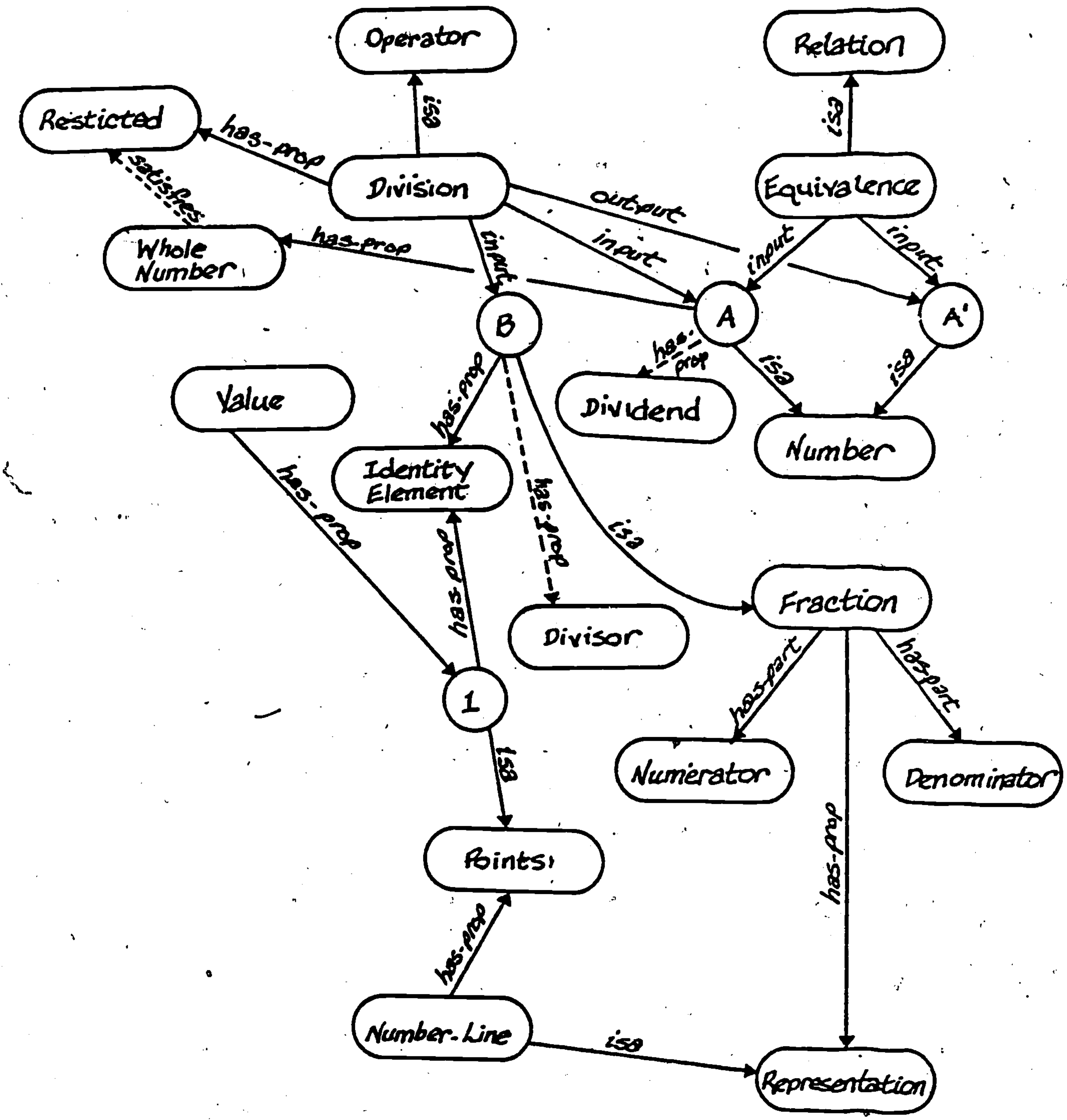


Figure 6: Konrad- Division/Equivalence

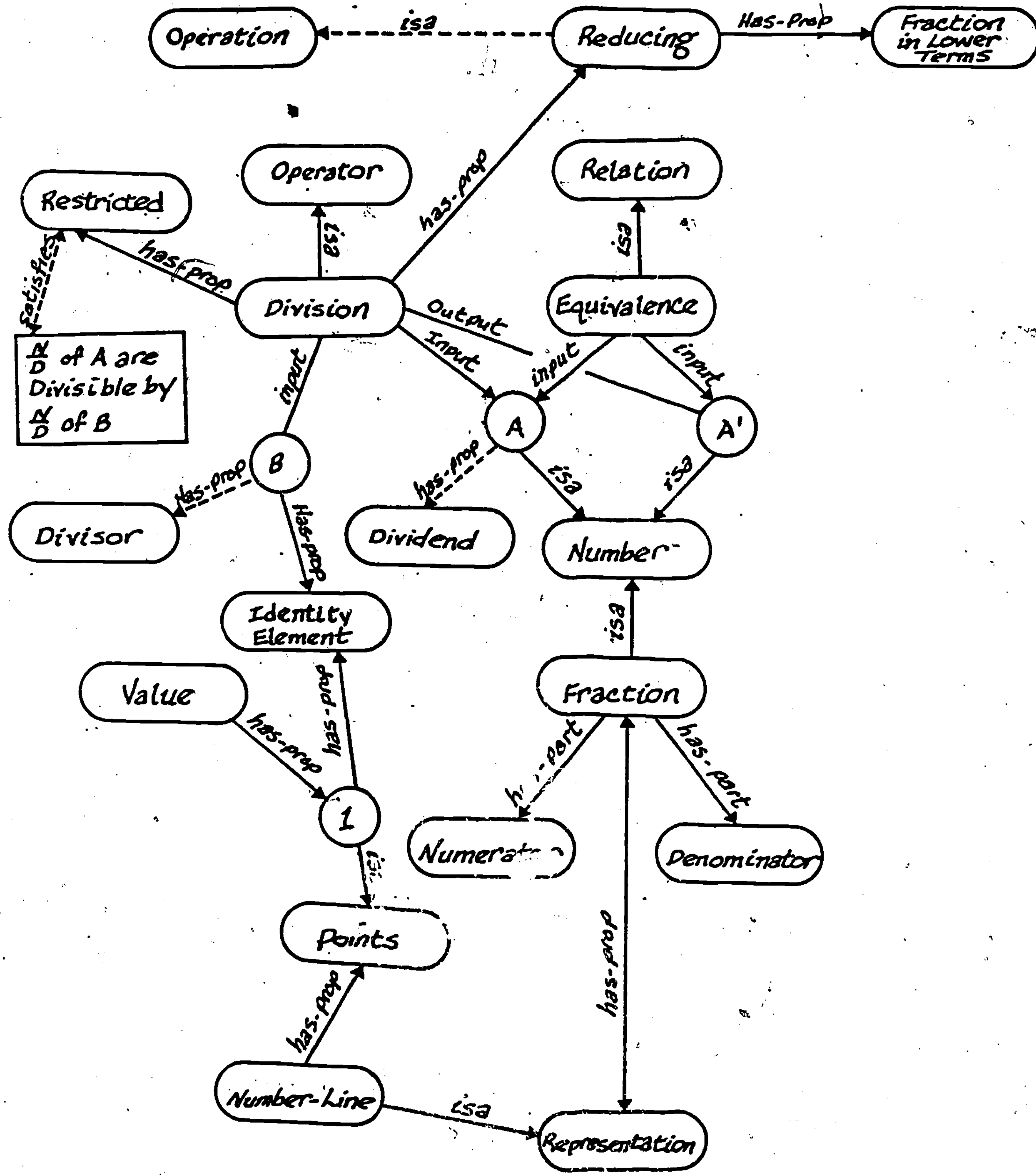


Figure 7: Konrad—Division/Reducing

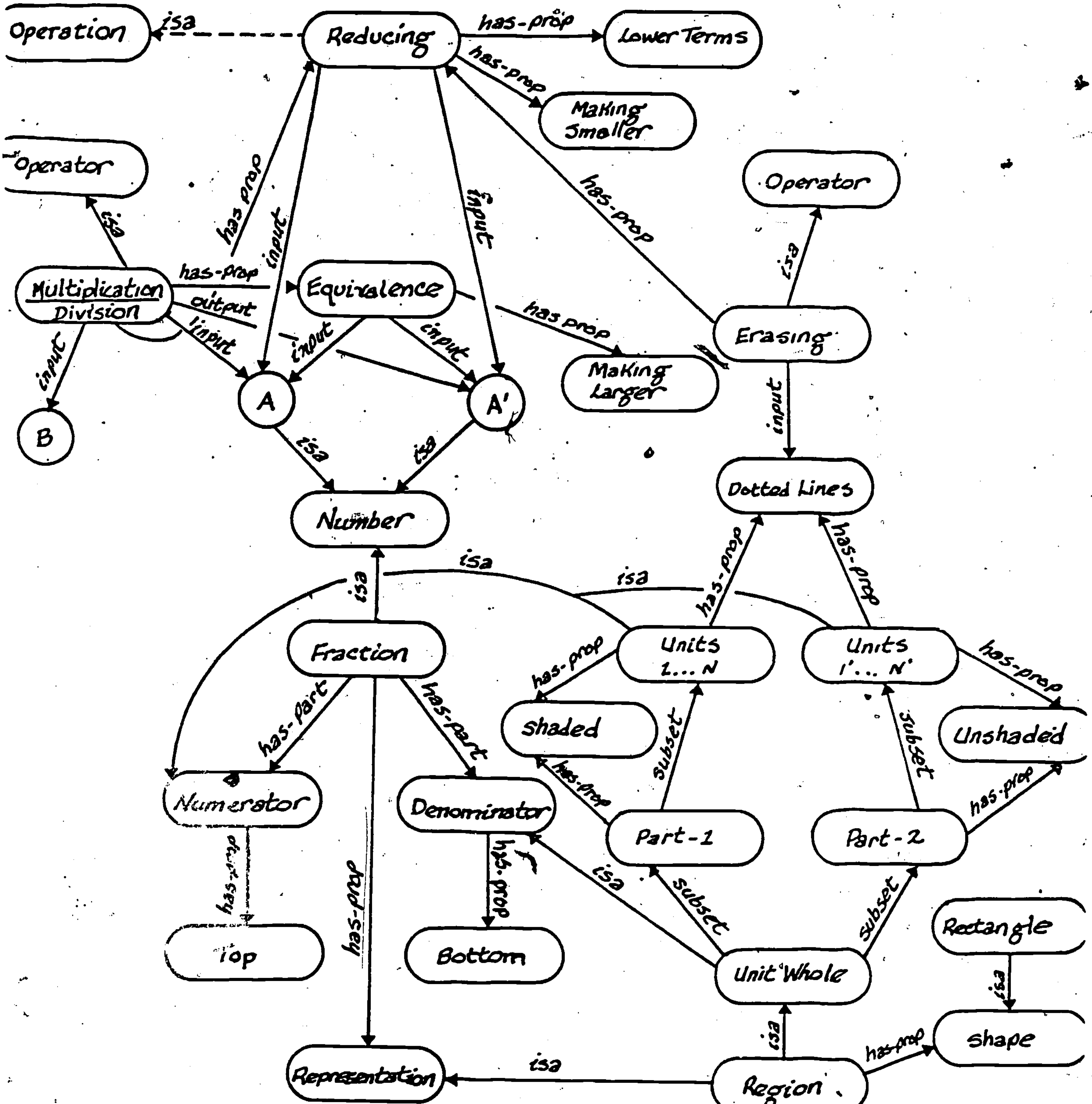


Figure 8: Generic Wall

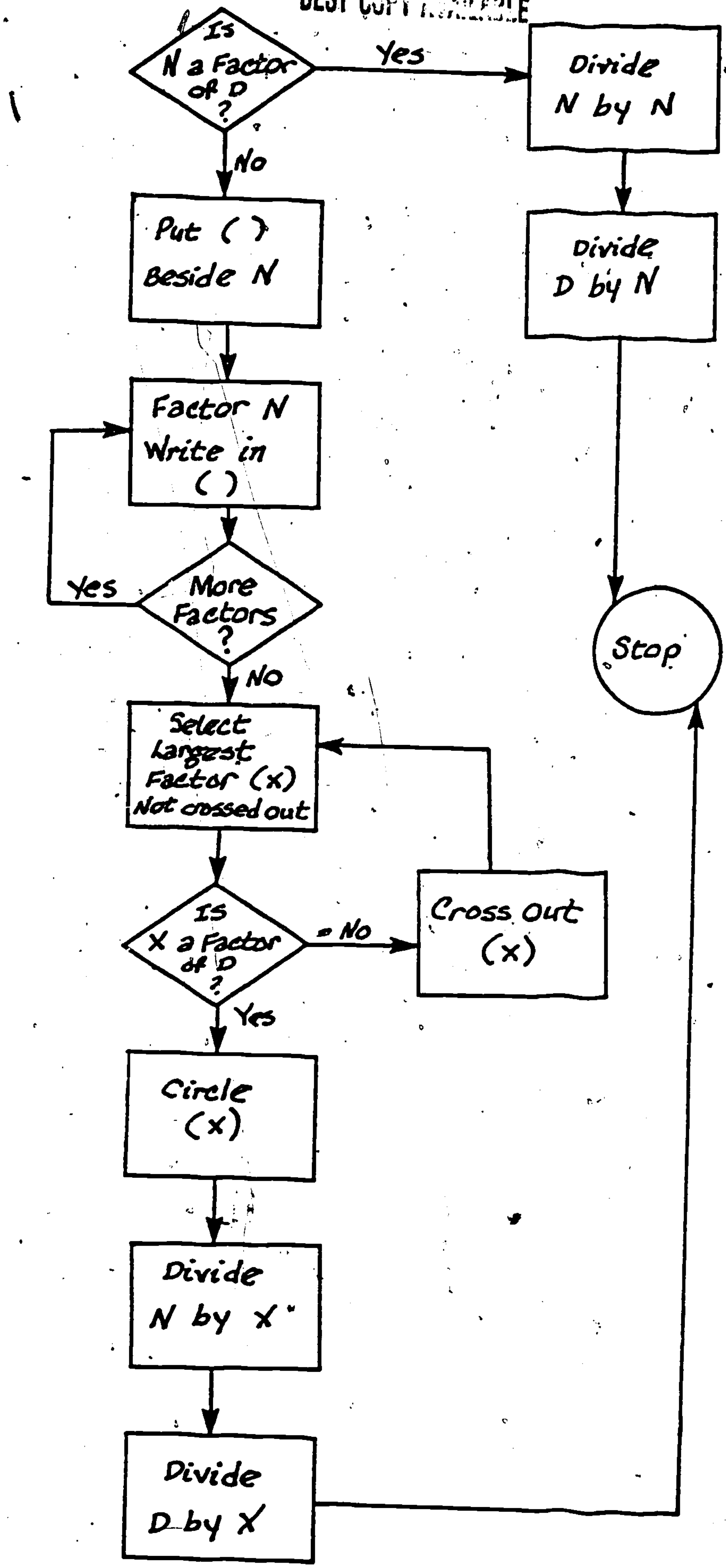


Figure 9: Wall Algorithm

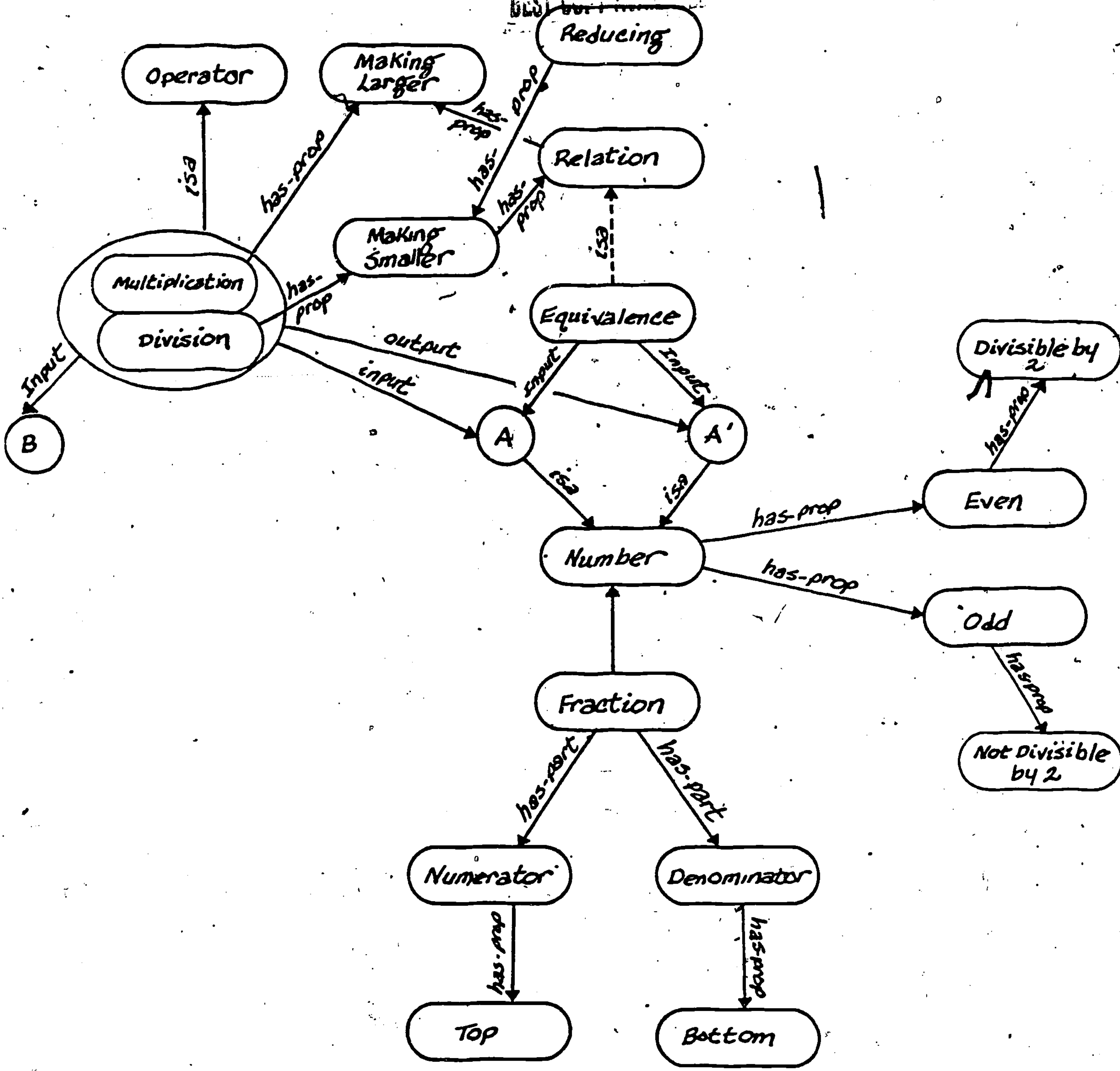


Figure 10: Generic Yoda

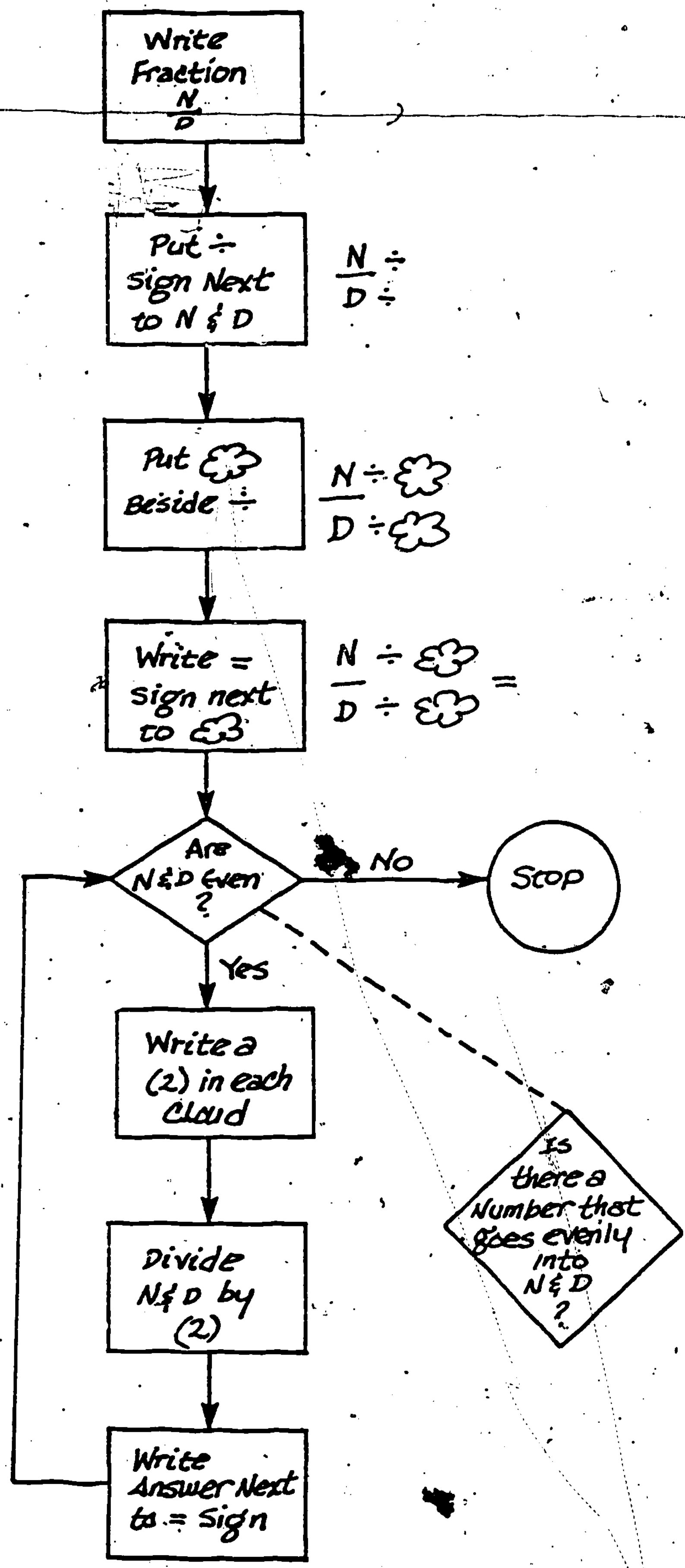
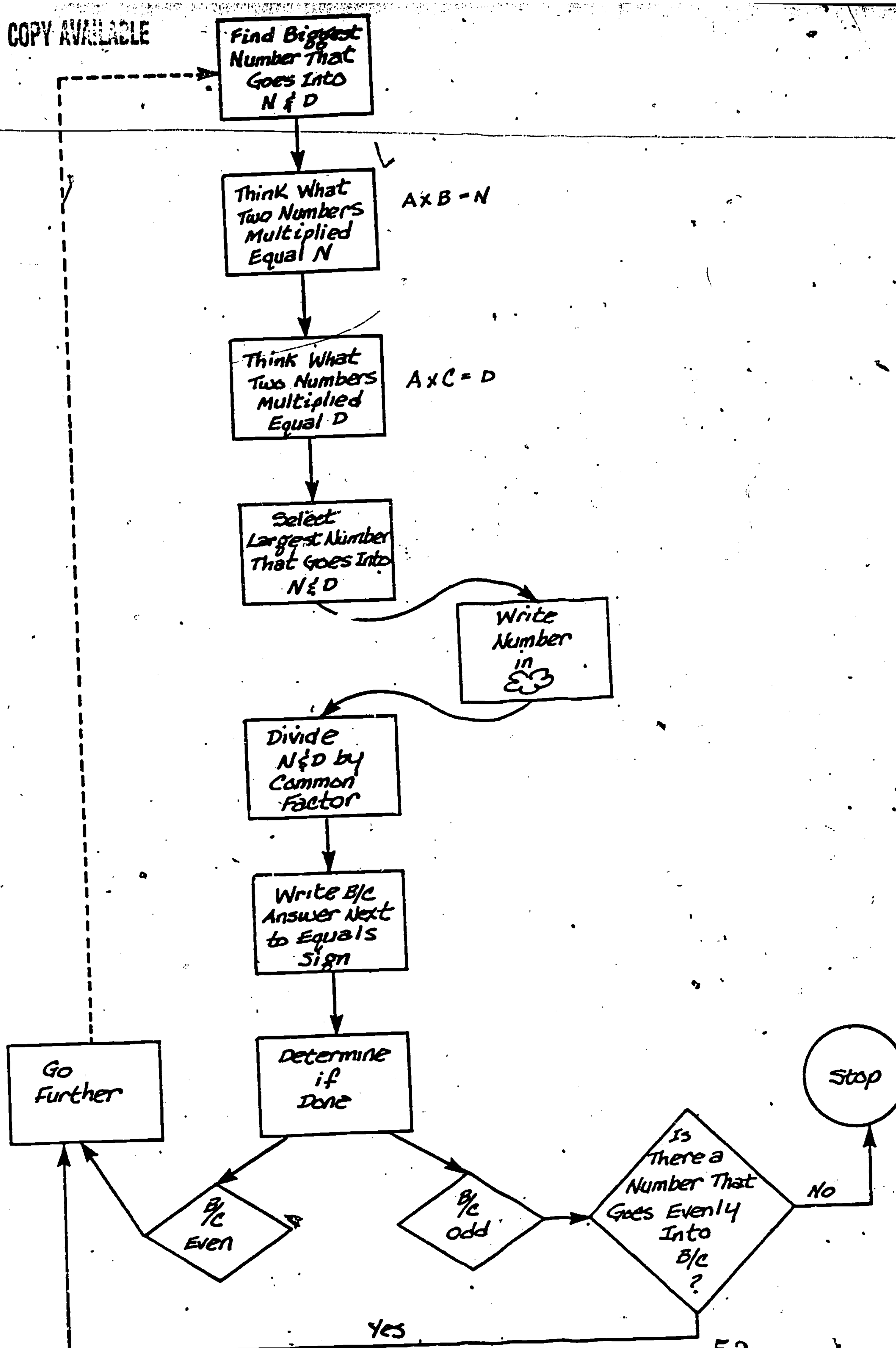


Figure 11 : Yoda Algorithm/1


Find Biggest Number That Goes Into N & D
Think What Two Numbers Multiplied Equal N
A x B = N
Think What Two Numbers Multiplied Equal D
A x C = D
Select Largest Number That Goes Into N & D
Write Number in
Divide N & D by Common Factor
Write B/C Answer Next to Equals Sign
Determine if Done
Go Further
B/C Even
B/C odd
Is There a Number That Goes Evenly Into B/C ?
No
Yes
Stop